18.06 Note Lecture 02

Elimination (Success & Failure)

$$\begin{aligned} x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \\ 4y + z &= 2 \end{aligned}$$

Solving $Ax = b$

$$A = \left[\begin{array}{ccc:c} \overset{1st\ pivot}{\boxed{1}} & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{(2,1)} \left[\begin{array}{ccc:c} \boxed{1} & 2 & 1 & 2 \\ 0 & \overset{2nd\ pivot}{\boxed{2}} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{(3,2)} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \overset{3rd\ pivot}{\boxed{5}} \end{bmatrix}$$

Pivot cannot be 0.

The last matrix $\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$ called matrix U.

Back-Substitution

$$\left[\begin{array}{ccc:c} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 0 & \boxed{5} & -10 \end{array}\right]$$

written as

$$\begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 6 \\ 5z &= -10 \end{aligned}$$

Those are the equations $Ux = c$. Obviously get $z = -2, y = 1, x = 2$.

Elimination Matrices

Big Picture

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3 \times \text{col } 1 + 4 \times \text{col } 2 + 5 \times \text{col } 3$$

Matrix $\times$ column $=$ is linear combination of columns of matrix.

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = \begin{matrix} 1 \times \text{row } 1 \\ + \\ 2 \times \text{row } 2 \\ + \\ 7 \times \text{row } 3 \end{matrix}$$

Row $\times$ matrix is linear combination of rows of matrix.

Elimination Step

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 1. Subtract $3\times$ row1 from row2

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ called **Elementary Matrix** E, and put the indexes E_{21} because that we needed to fix the $2\ 1$ position.

Step 2. Subtract $2\times$row2 from row3

$$\underbrace{\begin{bmatrix}1&0&1\\0&1&0\\0&-2&1\end{bmatrix}}_{E_{32}}\begin{bmatrix}1&2&1\\0&2&-2\\0&4&1\end{bmatrix}=\begin{bmatrix}1&2&1\\0&2&-2\\0&0&5\end{bmatrix}$$

Put steps together

$$E_{32}(E_{21}A)=U$$

because of **Associative Law** $(AB)C=A(BC)$, we can multiply the Es first

$$(E_{32}E_{21})A=U$$

and that gives the matrix that does everything at once.

Permutation Matrix

Exchange rows 1 and 2?

Assume a matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$. Suppose we want to exchanges those rows

$$\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{P}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}c&d\\a&b\end{bmatrix}$$

If we want to exchanges those columns

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{P}=\begin{bmatrix}b&a\\d&c\end{bmatrix}$$

These equations give us another fact that $AB\neq BA$.

According to the above conclusions, we can get a single matrix that does elimination by multiply E_{32} and E_{21} together.

But there is a better way to do this. The better way is to think not how to do I get A to U, but how do I get from U back to A?

Inverse Matrix

We want to find the matrix that undoes step 1

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

The matrix E^{-1} is the matrix that we needed.